Coarse embeddings of symmetric spaces and euclidean buildings

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The spaces considered are of the form:

$$X = \mathbb{R}^n \times S \times B$$

- S = symmetric space of non-compact type

  e.g: $\mathbb{H}^n$, $SL_n(\mathbb{R})/SO_n(\mathbb{R})$, G/K

- B = Euclidean building

  e.g: Trees, Bruhat Tits buildings of $SL_n(\mathbb{Q}_p)$

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Obstructions for coarse embeddings between such spaces?

Def 1 : X, Y metric spaces

$f : X \to Y$ is a **coarse embedding** if

$\exists\, \rho^-, \rho^+ : \mathbb{R}_+ \to \mathbb{R}_+$, $\rho^- \xrightarrow[+\infty]{} +\infty$ s.t

$\forall x_1, x_2 \in X$,

$$\rho^-\big(d(x_1,x_2)\big) \leqslant d\big(f(x_1), f(x_2)\big) \leqslant \rho^+\big(d(x_1,x_2)\big)$$

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Def 2 :

$X = \mathbb{R}^n \times S \times B$, The rank of X

$$rk(X) := \max \left\{ k \in \mathbb{N} \,/\, \exists\ \mathbb{R}^k \xhookrightarrow{\text{isom}} X \right\}$$

The Q.I case:

Thm [Anderson-Schroeder '86, Kleiner '98]

$X = \mathbb{R}^n \times S \times B$, $Y = \mathbb{R}^{n'} \times S' \times B'$

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Question : What about coarse embeddings?

Counter-example :

$$\mathbb{R}^n \xhookrightarrow{C.E} \mathbb{H}^{n+1} \text{ as a horosphere}$$

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$X = S \times B$, $Y = \mathbb{R}^n \times S' \times B'$

- if $rk(X) > rk(Y)$

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  Then $X \times \mathbb{R} \overset{C.E}{\not\hookrightarrow} Y$

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e.g :

- $G/K \overset{C.E}{\not\hookrightarrow} \mathbb{H}^n \times \mathbb{H}^p$, when $rk(G) \geq 3$
- $\underbrace{T_3 \times \cdots \times T_3}_{n} \times \mathbb{R} \overset{C.E}{\not\hookrightarrow} G/K$, $rk(G) \leq n$

Thank you!